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Flat time-like submanifolds in $S_{2q}^{2n-1}(1)$

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Abstract

In this paper it is shown that the $G_{n-q,n-q}^{q,q}$-II system gives the Gauss–Codazzi–Ricci equations of a class of flat time-like n-submanifolds with index q in $S_{2q}^{2n-1}(1)$, where $G_{n-q,n-q}^{q,q} = O(2n-2q,2q)/O(n-q,q) \times O(n-q,q)$ and $0 < q < n$. Moreover, we construct a dressing action on the $G_{n-q,n-q}^{q,q}$-system space of solutions of the $G_{n-q,n-q}^{q,q}$-system which gives rise to Bäcklund transformations for flat time-like n-submanifolds with index q in $S_{2q}^{2n-1}(1)$.

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1. Introduction

The classical Bäcklund transformation (BT) describes the transformation between surfaces with constant negative curvature in Euclidean space E^3 [1, 2]. With the development of integrable theory, the BT has become an important method of finding new solutions of partial differential equations. At the same time, many authors have presented some geometric generalizations. In [2], Chern and Terng introduced W-congruence and discussed BT between affine minimal surfaces in affine geometry. In [3–5], Tenenblat and Terng considered the generalization in high-dimensional space forms $N^{2n-1}(c)$ and obtained the generalized sine-Gordon and wave equation. On the other hand, the pseudo-Riemannian geometry has been a subject of wide interest [7]. In Lorentzian space forms $N_1^3(c)$, the generalization was considered in [8–12]. In [13], we have obtained BTs for Lorentzian n-submanifolds in $N_1^{2n-1}(c)$. The aim of this paper is to study the BT for flat time-like n-submanifolds with index q in $S_{2q}^{2n-1}(1)$ by using the $G_{n-q,n-q}^{q,q}$-system (see below) for $0 < q < n$, where $G_{n-q,n-q}^{q,q} = O(2n-2q,2q)/O(n-q,q) \times O(n-q,q)$. This paper is organized as follows: firstly we discuss the relation between the $G_{n-q,n-q}^{q,q}$-system and a class of flat time-like n-submanifolds with index q in $S_{2q}^{2n-1}(1)$. Afterwards we construct a dressing action on the space of solutions of the $G_{n-q,n-q}^{q,q}$-system. Finally, it is shown that the dressing action gives rise to BTs for flat time-like n-submanifolds with index q in $S_{2q}^{2n-1}(1)$.

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2. Flat time-like submanifolds in $S^{2n-1}_{2q}(1)$ associated with $G^{q,q}_{n-q,n-q}$-system

Recently Terng *et al* [16, 17], Ferus and Pedit [18] established a beautiful relation between integrable system and submanifold geometry. They found that the submanifolds in a certain symmetric space whose Gauss–Codazzi–Ricci equations are given by a nonlinear first-order system, the U/K-system, which is putting the n first flows of ZS-AKNS together. This means finding submanifolds M in a certain symmetric space whose Gauss–Codazzi–Ricci equations are equivalent to the U/K-system. The direct approach may provide ways of finding Lax pairs for submanifolds M. Terng *et al* [16, 17] carried out the project for the real Grassmannian manifolds of space-like m-dimensional linear subspaces in R^{m+n} and in $R^{m+n,1}$. To be self-contained, below we give a short review of some known facts about the $G^{q,q}_{n-q,n-q}$-system, see [16, 17] for details. Let

$$\mathcal{U} = o(2n-2q, 2q) = \left\{ X \in gl(n+n) | X^t \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix} + \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix} X = 0 \right\}$$

and $\sigma_* : \mathcal{U} \to \mathcal{U}$ be an involution defined by $\sigma_*(X) = \tilde{I}^{-1}_{n,n} X \tilde{I}_{n,n}$, where $J = \tilde{I}_{q,n-q} = \mathrm{diag}(\epsilon_1, \ldots, \epsilon_n)$ and $\tilde{I}_{q,p} = \begin{pmatrix} -I_q & 0 \\ 0 & I_p \end{pmatrix}$. Then the Cartan decomposition is $\mathcal{U} = \mathcal{K} + \mathcal{P}$, where

$$\mathcal{K} = o(n-q, q) \times o(n-q, q) = \left\{ \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} \middle| X_1, X_2 \in o(n-q, q) \right\}$$

$$\mathcal{P} = \left\{ \begin{pmatrix} 0 & F \\ -JF^tJ & 0 \end{pmatrix} \middle| F \in \mathcal{M}_{n\times n} \right\}.$$

Here $\mathcal{M}_{n\times n}$ is the set of $n \times n$ matrices. It is easy to see that

$$\mathcal{A} = \left\{ \begin{pmatrix} 0 & C \\ -JC^tJ & 0 \end{pmatrix} \middle| C \in \mathcal{M}_{n\times n}, c_{ij} = 0 \text{ if } i \neq j \right\}$$

is a maximal Abelian subalgebra in $\mathcal{P}$. Let

$$a_i = \begin{pmatrix} 0 & C_i \\ -JC_i^tJ & 0 \end{pmatrix}$$

where $C_i \in \mathcal{M}_{n\times n}$ is the matrix whose entries are all zero except the (i, i)th entry that is equal to 1. Then $a_1, \ldots, a_n$ form a basis of $\mathcal{A}$ and

$$\mathcal{P} \cap \mathcal{A}^\perp = \left\{ \begin{pmatrix} 0 & F \\ -JF^tJ & 0 \end{pmatrix} \middle| F \in \mathcal{M}_{n\times n}, f_{ii} = 0, \text{ for } 1 \leqslant i \leqslant n \right\}.$$

The $G^{q,q}_{n-q,n-q}$-*system*, according to the terminology of [16], is the following PDE for $F = (f_{ij}) : R^n \to gl(n)_*$ such that

$$\theta_\lambda = \sum_{i=1}^{n} \left\{ \lambda \begin{pmatrix} 0 & C_i \\ -JC_i^tJ & 0 \end{pmatrix} + \begin{pmatrix} JC_iF^tJ - FC_i{}^t & 0 \\ 0 & C_i^tF - JF^tC_iJ \end{pmatrix} \right\} \mathrm{d}x_i \tag{2.1}$$

is a family of flat connections on R^n for all $\lambda \in \mathbf{C}$, i.e.

$$\mathrm{d}\theta_\lambda + \theta_\lambda \wedge \theta_\lambda = 0 \tag{2.2}$$

where $gl(n)_* = \{(x_{ij}) \in gl(n) | x_{ii} = 0, 1 \leqslant i \leqslant n\}$. Hence there exists a smooth map $E : R^n \times \mathbf{C} \to O(2n-2q, 2q)$ such that $E^{-1}\,\mathrm{d}E = \theta_\lambda$, $E(0, \lambda) = I$ and E is often called the frame or trivialization of the flat connection θ_λ.

The $G^{q,q}_{n-q,n-q}$-reality condition is

$$\begin{cases} \overline{g(\bar\lambda)} = g(\lambda) \\ \tilde I_{n,n} g(\lambda) \tilde I_{n,n} = g(-\lambda) \\ g(\lambda)\begin{pmatrix} J & 0 \\ 0 & J\end{pmatrix} g(\lambda)^t = \begin{pmatrix} J & 0 \\ 0 & J\end{pmatrix}. \end{cases} \tag{2.3}$$

Let $g = \begin{pmatrix} B & 0 \\ 0 & A\end{pmatrix} \in O(n-q,q) \times O(n-q,q)$ be a solution of $g^{-1}\,\mathrm{d}g = \theta_0$ and $h = \begin{pmatrix} I_n & 0 \\ 0 & A\end{pmatrix}$. Make the gauge transformation of θ_λ by h,

$$\Omega_\lambda = h * \theta_\lambda = h\theta_\lambda h^{-1} - \mathrm{d}h h^{-1} = \sum_{i=1}^n \left\{\begin{pmatrix} JC_iF^tJ - FC_i{}^t & -JC_iA^tJ\lambda \\ AC_i^t\lambda & 0\end{pmatrix}\right\} \mathrm{d}x_i \tag{2.4}$$

which is also a family of flat connections on R^n for all $\lambda \in \mathbf{C}$, i.e.,

$$\begin{cases} \epsilon_i (f_{ij})_{x_i} + \epsilon_j (f_{ji})_{x_j} + \sum_{k=1}^n \epsilon_k f_{ki} f_{kj} = 0 & \text{if } \ i \neq j, \\ (f_{ij})_{x_k} = f_{ik} f_{kj} & \text{if } \ i, j, k \text{ are distinct} \\ (a_{ij})_{x_k} = a_{ik} f_{kj} & \text{if } \ j \neq k \end{cases} \tag{2.5}$$

where $A = (a_{ij})$, $F = (f_{ij})$. Note that Eh^{-1} is the frame of Ω_λ. We call (2.5) the *$G^{q,q}_{n-q,n-q}$-II system.*

Example 2.1. Consider the case $n = 2$ and $q = 1$. By choosing

$$A = \begin{pmatrix} \cosh\frac{u}{2} & \sinh\frac{u}{2} \\ \sinh\frac{u}{2} & \cosh\frac{u}{2}\end{pmatrix}$$

where u is a differentiable function of x_1, x_2, the equation $G^{q,q}_{n-q,n-q}$-system II (2.5) reduces to $u_{x_1x_1} - u_{x_2x_2} = 0$ which is the homogeneous wave equation. Hence $G^{q,q}_{n-q,n-q}$-system II (2.5) may be regarded as the generalized wave equation associated with the non-compact symmetric space $G^{q,q}_{n-q,n-q} (q \neq 0)$. In fact when $q = 0$, the $G_{n,n}$-system is the generalized wave equation associated with the Riemannian symmetric space $G_{n,n}$ in [5, 16] which has been studied by many authors (see [6] for details).

Lemma 2.2. *Let (A, F) be a solution of the $G^{q,q}_{n-q,n-q}$-II system* (2.5)*, then there exists an $O(2n-2q, 2q)$-valued $g_1 = (e_1, \ldots, e_{2n})$ such that $g_1^{-1}\,\mathrm{d}g_1 = \Omega_\lambda|_{\lambda=1}$. If all entries of the last row of A are nonzero then e_{2n} defines a local isometric immersion of a flat time-like n-submanifold in $S^{2n-1}_{2q}(1)$ with flat, non-degenerate normal bundle such that the two fundamental forms are*

$$I = \sum_{i=1}^n \epsilon_i a_{ni}^2\,\mathrm{d}x_i^2 \qquad II = \sum_{i=1}^n \sum_{l=1}^{n-1} \epsilon_k a_{ni} a_{li}\,\mathrm{d}x_i^2 e_{n+l} \tag{2.6}$$

where $A = (a_{ij})$ and $\{x_1, \ldots, x_n\}$ are often called line of curvature coordinates.

Proof. Write $\Omega_\lambda|_{\lambda=1} = \left(\Omega_a^b\right)$. Set

$$\omega^i = \Omega_i^{2n} \qquad \omega_i^j = \Omega_j^i \qquad \omega_i^{n+k} = \Omega_{n+k}^i \qquad \omega_{n+k}^{n+l} = \Omega_{n+l}^{n+k} = 0$$
$$1 \leqslant i \quad j \leqslant n \quad 1 \leqslant k \quad l \leqslant n-1.$$

Since $\Omega_\lambda|_{\lambda=1} = \left(\Omega_a^b\right)$ is flat, we have $\mathrm{d}\omega^i = \sum_{j=1}^n \omega^j \wedge \omega_j^i$. Since the last row of A are nonzero, then $\left(\omega_i^j\right)_{nn}$ is the Levi-Civita connection 1-form for the metric $\sum_{i=1}^n \epsilon_i (\omega^i)^2$. The rest follows from $g_1^{-1}\,\mathrm{d}g_1 = \Omega_\lambda|_{\lambda=1}$ and the fundamental theorem of submanifolds in the pseudo-Riemannian space [7]. □

Remark 2.3. Note that the converse is not true. In general, a flat time-like n-submanifold in $S^{2n-1}_{2q}(1)$ has a flat and non-degenerate normal bundle which could not assure the existence of the above line of curvature coordinates.

3. Loop group actions and BTs

In this section we assume that the flat time-like n-submanifolds in $S^{2n-1}_{2q}(1)$ have flat and non-degenerate normal bundle, and admit the above line of curvature coordinates. We shall show that a dressing action on the space of solutions of the $G^{q,q}_{n-q,n-q}$-system (2.2) gives rise to BTs for flat time-like n-submanifolds in $S^{2n-1}_{2q}(1)$.

3.1. Loop group actions for $G^{q,q}_{n-q,n-q}$-system

Define a rational map with only a simple pole

$$K_{s,\beta}(\lambda) = \frac{1}{\lambda - \mathrm{i}s}\begin{pmatrix} s\beta & \lambda I_n \\ -\lambda I_n & sJ\beta^t J \end{pmatrix} \tag{3.1}$$

where $s \in R$ is a nonzero constant and $\beta \in O(n-q,q)$. It is easily verified that $g_{s,\beta}(\lambda) = \frac{\lambda - \mathrm{i}s}{\sqrt{\lambda^2+s^2}} K_{s,\beta}$ satisfies the $G^{q,q}_{n-q,n-q}$-reality condition (2.3). We call $K_{s,\beta}(\lambda)$ a simple element of the $G^{q,q}_{n-q,n-q}$-system (2.2) according to the terminology of [14, 15]. By the method of Terng and Uhlenbeck [15], one knows that if E is a frame of a solution F of the $G^{q,q}_{n-q,n-q}$-system (2.2), then $K_{s,\beta}E$ can be factored as $\tilde{E}K_{s,\tilde{\beta}(x)}$ for some functions $\tilde{E}$ and $K_{s,\tilde{\beta}(x)}$.

Theorem 3.1. *Let F be a solution of the $G^{q,q}_{n-q,n-q}$-system* (2.2) *and E a frame of F. Write $E(x,-\mathrm{i}s) = \begin{pmatrix} \eta_1 & \eta_2 \\ \eta_3 & \eta_4 \end{pmatrix}$ with $\eta_i \in gl(n)$ and set*

$$\tilde{\beta} = (\mathrm{i}\eta_4 - \beta\eta_2)^{-1}(\mathrm{i}\beta\eta_1 + \eta_3) \qquad \tilde{E} = K_{s,\beta}E(x,\lambda)K^{-1}_{s,\tilde{\beta}(x)}(\lambda). \tag{3.2}$$

Then

(1) $\tilde{F} = K_{s,\beta}\#F = JF^tJ + s\tilde{\beta}_*$ is a solution of the $G^{q,q}_{n-q,n-q}$-system (2.2) *and $\tilde{E}$ is a frame of $\tilde{F}$, where $\tilde{\beta}_*$ is the matrix whose ijth entry is $\tilde{\beta}_{ij}$ for $i \neq j$ and is 0 for $i = j$.*

(2) $\tilde{\beta}$ is a solution of

$$\begin{cases} \mathrm{d}\tilde{\beta} = \tilde{\beta}(J\delta F^tJ - F\delta) + (JF^t\delta J - \delta F)\tilde{\beta} - s\delta + s\tilde{\beta}\delta\tilde{\beta} \\ \tilde{\beta}^tJ\tilde{\beta} = J. \end{cases} \tag{3.3}$$

Proof. We first prove that $\tilde{E}(x,\lambda)$ is holomorphic in $\lambda \in C$. By (3.2), one knows that $\tilde{E}(x,\lambda)$ is holomorphic for all $\lambda \in C$ except at $\lambda = \pm\mathrm{i}s$ and has simple poles at $\mathrm{i}s$ and $-\mathrm{i}s$. One only needs to prove that the residues of $\tilde{E}$ are zero at both $\mathrm{i}s$ and $-\mathrm{i}s$. Note that

$$\mathrm{Res}(\tilde{E}, -\mathrm{i}s) = \frac{\mathrm{i}s}{2}\begin{pmatrix} \beta & -\mathrm{i}I_n \\ \mathrm{i}I_n & J\beta^tJ \end{pmatrix}\begin{pmatrix} \eta_1 & \eta_2 \\ \eta_3 & \eta_4 \end{pmatrix}\begin{pmatrix} J\tilde{\beta}^tJ & \mathrm{i}I_n \\ -\mathrm{i}I_n & \tilde{\beta} \end{pmatrix}.$$

Owing to (3.2) and $\beta \in O(n-q,q)$, one gets $\mathrm{Res}(\tilde{E}, -\mathrm{i}s) = 0$. By using the reality condition $\overline{E(x,-\mathrm{i}s)} = E(x,\mathrm{i}s)$, similarly one may show that $\mathrm{Res}(\tilde{E},\mathrm{i}s) = 0$. Hence $\tilde{E}(x,\lambda)$ is holomorphic in $\lambda \in C$. Let $\tilde{\theta}_\lambda = \tilde{E}^{-1}\,\mathrm{d}\tilde{E}$, then $\tilde{\theta}_\lambda$ is holomorphic for $\lambda \in C$. By using $\theta_\lambda = E^{-1}\,\mathrm{d}E$ and $\tilde{E} = K_{s,\beta}E(x,\lambda)K^{-1}_{s,\tilde{\beta}(x)}(\lambda)$, one gets $K_{s,\beta}\tilde{\theta}_\lambda = K_{s,\tilde{\beta}}\theta_\lambda - \mathrm{d}K_{s,\tilde{\beta}}$. Comparing the coefficient of λ^j $(j = 0, 1, 2)$, then one gets (1) and (2). This completes the proof of the theorem. □

Note that (3.3) is the BT of the $G_{n-q,n-q}^{q,q}$-system (2.2). Hence one has

Corollary 3.2. *Let F be a solution of the $G_{n-q,n-q}^{q,q}$-system* (2.2) *and E a frame of F. Then the system* (3.3) *is solvable for $\tilde{\beta}$. Moreover if $\tilde{\beta}$ is a solution of* (3.3) *with initial condition $\tilde{\beta}(0) = \beta$, then $\tilde{F} = JF^tJ + s\tilde{\beta}_*$ is a solution of the $G_{n-q,n-q}^{q,q}$-system* (2.2).

In order to get the permutability formula for the $G_{n-q,n-q}^{q,q}$-system (2.2), one may consider the relation among simple elements.

Theorem 3.3. *Let $s_1 \neq s_2 \in R$ be nonzero constants and $\beta_1, \beta_2 \in O(n-q, q)$ constant matrices. Set*

$$\alpha_1 = \left(s_1 I_n - s_2 J\beta_2^t\beta_1\right)(s_1\beta_1 - s_2\beta_2)^{-1} \qquad \alpha_2 = \left(s_2 I_n - s_1 J\beta_1^t\beta_2\right)(s_2\beta_2 - s_1\beta_1)^{-1}. \tag{3.4}$$

If (i) $\phi = s_1\beta_1 - s_2\beta_2$ is non-singular; (ii) $\beta_1^t J\beta_2 \neq \frac{s_2}{s_1}J$ and $\beta_2^t J\beta_1 \neq \frac{s_1}{s_2}J$. Then

(1) $K_{s_1,\alpha_1} \circ K_{s_2,\beta_2} = K_{s_2,\alpha_2} \circ K_{s_1,\beta_1}$, where $\alpha_1, \alpha_2 \in O(n-q, q)$;
(2) If $K_{s_1,\alpha_1} \circ K_{s_2,\beta_2} = K_{s_2,\alpha_2} \circ K_{s_1,\beta_1}$, then β_1, β_2 and α_1, α_2 are related as in (3.4).

Proof. It is easily verified that $J\alpha_i^{-1} = \alpha_i^t J$, hence $\alpha_i \in O(n-q, q)$ for $i = 1, 2$. Note that $K_{s_1,\alpha_1} \circ K_{s_2,\beta_2} = K_{s_2,\alpha_2} \circ K_{s_1,\beta_1}$ is equivalent to

$$\begin{pmatrix} s_1\alpha_1 & \lambda I_n \\ -\lambda I_n & s_1 J\alpha_1^t J \end{pmatrix}\begin{pmatrix} s_2\beta_2 & \lambda I_n \\ -\lambda I_n & s_2 J\beta_2^t J \end{pmatrix} = \begin{pmatrix} s_2\alpha_2 & \lambda I_n \\ -\lambda I_n & s_2 J\alpha_2^t J \end{pmatrix}\begin{pmatrix} s_1\beta_1 & \lambda I_n \\ -\lambda I_n & s_1 J\beta_1^t J \end{pmatrix}$$

that is,

$$\begin{aligned} \alpha_1\beta_2 &= \alpha_2\beta_1 & s_1\alpha_1 + s_2 J\beta_2^t J &= s_2\alpha_2 + s_1 J\beta_1^t J \\ \beta_2\alpha_1 &= \beta_1\alpha_2 & s_2\beta_2 + s_1 J\alpha_1^t J &= s_1\beta_1 + s_2 J\alpha_2^t J. \end{aligned}$$

By using (i), (ii) and (3.4), one knows that the theorem holds. □

As a consequence of the theorem, one obtains the permutability formula.

Corollary 3.4. *Let s_i, β_i, α_i for $i = 1, 2$ as in theorem 3.3, F be a solution of the $G_{n-q,n-q}^{q,q}$-system* (2.2) *and $F_i = K_{s_i,\beta_i}\#F = JF^tJ + s_i\tilde{\beta}_{i*}$ for $i = 1, 2$ as given in theorem 3.1. Then the permutability formula is*

$$\begin{aligned} F_3 &= \left(K_{s_1,\alpha_1} \circ K_{s_2,\beta_2}\right)\#F = JF^tJ + s_1\tilde{\alpha}_{1*} + s_2\tilde{\beta}_{2*} \\ &= \left(K_{s_2,\alpha_2} \circ K_{s_1,\beta_1}\right)\#F = JF^tJ + s_2\tilde{\alpha}_{2*} + s_1\tilde{\beta}_{1*}. \end{aligned} \tag{3.5}$$

3.2. BTs for flat time-like n-submanifolds in $S_{2q}^{2n-1}(1)$

Firstly we give a geometric definition on the BT for flat time-like n-submanifolds in $S_{2q}^{2n-1}(1)$.

Definition 3.5. *Let M and $\tilde{M}$ be time-like submanifolds in $S_{2q}^{2n-1}(1)$ with flat, non-degenerate normal bundles. A diffeomorphism $\mathcal{L} : M \to \tilde{M}$ is a BT with constant ρ if there exist local pseudo-orthonormal O(2n-2q, 2q)-frames $\{e_{2n}, e_A\}$ and $\{\tilde{e}_{2n}, \tilde{e}_A\}$ of M and $\tilde{M}$, respectively, such that*

(1) e_{2n} and $\tilde{e}_{2n}$ are time-like immersions in $S_{2q}^{2n-1}(1)$;

(2) $\{e_\alpha\}$ *and* $\{\tilde{e}_\alpha\}$ *are parallel normal frames for* M *and* $\tilde{M}$*, respectively;*

(3) $(\tilde{e}_1, \ldots, \tilde{e}_{2n}) = (e_1, \ldots, e_{2n})\begin{pmatrix} \cos\rho I_n & -\sin\rho I_n \\ \sin\rho I_n & \cos\rho I_n \end{pmatrix}$ *for all* $x \in M$*.*

By lemma 2.2, one knows that solutions of the $G^{q,q}_{n-q,n-q}$-II system correspond to flat time-like n-submanifolds with flat, non-degenerate normal bundle in $S^{2n-1}_{2q}(1)$. Note that the $G^{q,q}_{n-q,n-q}$-II system (2.5) is gauge equivalent to the $G^{q,q}_{n-q,n-q}$-system (2.2). Hence the immersion f has a Lax pair

$$E^{-1}\,\mathrm{d}E = \theta_\lambda = \begin{pmatrix} J\delta F^t J - F\delta & \delta\lambda \\ -J\delta J\lambda & \delta F - JF^t\delta J \end{pmatrix} \tag{3.6}$$

where $\delta = \mathrm{diag}(\mathrm{d}x_1, \ldots, \mathrm{d}x_n)$. Hence theorems 3.1, 3.3 and corollary 3.4 give a method of constructing a new flat time-like n-submanifold in $S^{2n-1}_{2q}(1)$ from a given one. Geometrically this gives the geometric transformation which is the BT defined by definition 3.5, and the analogue of classical Bianchi theorem (see theorem 3.6 and corollary 3.7).

Theorem 3.6. *Let* $F, E, K_{s,\beta}, \tilde{\beta}, \tilde{E}, \tilde{F}$ *as in theorem 3.1. Write*

$$E(x,0) = \begin{pmatrix} B(x) & 0 \\ 0 & A(x) \end{pmatrix} \qquad \tilde{E}(x,0) = \begin{pmatrix} \tilde{B}(x) & 0 \\ 0 & \tilde{A}(x) \end{pmatrix}. \tag{3.7}$$

Let

$$\begin{aligned} N(x) &= E^{II}(x,1) = E(x,1)\begin{pmatrix} I_n & 0 \\ 0 & A^{-1}(x) \end{pmatrix} \\ \tilde{N}(x) &= g^{-1}_{s,\beta}(1)\tilde{E}(x,1)\begin{pmatrix} I_n & 0 \\ 0 & (A\tilde{\beta})^{-1}(x) \end{pmatrix} = N(x)\begin{pmatrix} \cos\rho J\tilde{\beta}^t J & -\sin\rho(A\tilde{\beta})^{-1} \\ \sin\rho A & \cos\rho I_n \end{pmatrix} \end{aligned} \tag{3.8}$$

where $\rho = \arctan\frac{1}{s}$*. Let* e_i *and* $\tilde{e}_i$ *for* $1 \leqslant i \leqslant 2n$ *denote the ith column of N(x) and* $\tilde{N}(x)$*, respectively. Then* $\mathcal{L} : e_{2n}(x) \to \tilde{e}_{2n}(x)$ *is the BT for flat time-like n-submanifolds with constant* ρ *defined by definition 3.5.*

Proof. Since $\beta \in O(n-q,q)$ is a constant matrix, $\begin{pmatrix} J\beta^t J & 0 \\ 0 & \beta \end{pmatrix}\tilde{E}(x,0)$ is also a frame of the flat connection $\tilde{\theta}_\lambda$ as in the form of (2.1) for $\tilde{F}$ at $\lambda = 0$. By using (2.4), (3.8) and theorem 3.1, one may know (A, F) and $(\tilde{A}, \tilde{F})$ are solutions of the $G^{q,q}_{n-q,n-q}$-II system (2.5) and $N, \tilde{N}$ are the corresponding frames at $\lambda = 1$, respectively. It follows from lemma 2.1 that $e_{2n}, \tilde{e}_{2n}$ are flat time-like n-submanifolds with flat, non-degenerate normal bundle in $S^{2n-1}_{2q}(1)$. $\{e_{n+1}, \ldots, e_{2n-1}\}$ and $\{\tilde{e}_{n+1}, \ldots, \tilde{e}_{2n-1}\}$ are parallel normal frames for e_{2n} and $\tilde{e}_{2n}$, respectively. Let

$$N_A = N\begin{pmatrix} (A\tilde{\beta})^{-1} & 0 \\ 0 & I_n \end{pmatrix} \qquad \tilde{N}_A = \tilde{N}\begin{pmatrix} A^{-1} & 0 \\ 0 & I_n \end{pmatrix}$$

then we have

$$\tilde{N}_A = N\begin{pmatrix} \cos\rho J\tilde{\beta}^t J & -\sin\rho(A\tilde{\beta})^{-1} \\ \sin\rho A & \cos\rho I_n \end{pmatrix}\begin{pmatrix} A^{-1} & 0 \\ 0 & I_n \end{pmatrix} = N_A\begin{pmatrix} \cos\rho I_n & -\sin\rho I_n \\ \sin\rho I_n & \cos\rho I_n \end{pmatrix}. \tag{3.9}$$

The last n column vectors of N and N_A, $\tilde{N}$ and $\tilde{N}_A$ are the same and they are normal frames. Geometrically, (3.9) is the BT as in definition 3.5. □

Using theorem 3.6 and corollary 3.4, one gets the following analogue of the Bianchi permutability formula.

Corollary 3.7. *Let $\mathcal{L}_i : M \to M_i$ be BTs for flat time-like n-submanifolds with flat, non-degenerate normal bundle in $S^{2n-1}_{2q}(1)$ corresponding to the solution of K_{s_i,β_i} for $i = 1, 2$. If $s_1 \neq s_2 \in R$ are nonzero constants, $\beta_1, \beta_2 \in O(n-q, q)$ constant matrices and $s_1\beta_1 - s_2\beta_2$ is non-singular, $\beta_1^t J\beta_2 \neq \frac{s_2}{s_1} J$ and $\beta_2^t J\beta_1 \neq \frac{s_1}{s_2} J$. Then there exists a unique flat time-like n-submanifold M_3 with flat, non-degenerate normal bundle in $S^{2n-1}_{2q}(1)$ and BTs $\tilde{\mathcal{L}}_1 : M_2 \to M_3$, $\tilde{\mathcal{L}}_2 : M_1 \to M_3$ such that $\tilde{\mathcal{L}}_1 \circ \mathcal{L}_2 = \tilde{\mathcal{L}}_2 \circ \mathcal{L}_1$.*

Finally we apply theorems 3.1 and 3.6 to construct an explicit immersion of flat time-like n-submanifold in S^{2n-1}_{2q} with flat and non-degenerate normal bundle.

Example 3.8. Choose the vacuum solution $F = 0$. The corresponding flat connection θ_λ of $G^{q,q}_{n-q,n-q}$-system (2.1) is

$$E^{-1}\,\mathrm{d}E = \theta_\lambda = \begin{pmatrix} 0 & \delta\lambda \\ -J\delta J\lambda & 0 \end{pmatrix} \qquad E(0,\lambda) = I_{2n}. \tag{3.10}$$

Hence one may get

$$E(x,\lambda) = \begin{pmatrix} E_{11}(x,\lambda) & E_{12}(x,\lambda) \\ E_{21}(x,\lambda) & E_{22}(x,\lambda) \end{pmatrix}$$

where

$$\begin{aligned} E_{11}(x,\lambda) &= E_{22}(x,\lambda) = \mathrm{diag}(\cos(\lambda x_1), \ldots, \cos(\lambda x_n)) \\ E_{12}(x,\lambda) &= -E_{21}(x,\lambda) = \mathrm{diag}(\sin(\lambda x_1), \ldots, \sin(\lambda x_n)). \end{aligned} \tag{3.11}$$

By using (3.2), one obtains $\tilde{\beta} = (\mathrm{i}\eta_4 - \beta\eta_2)^{-1}(\mathrm{i}\beta\eta_1 + \eta_3)$, where

$$\begin{aligned} \eta_1 &= \eta_4 = \mathrm{diag}(\cosh(sx_1), \ldots, \cosh(sx_n)) \\ \eta_3 &= -\eta_2 = \mathrm{diag}(\mathrm{i}\sinh(sx_1), \ldots, \mathrm{i}\sinh(sx_n)). \end{aligned} \tag{3.12}$$

Note that $E(x,0) = I_{2n}$, hence $A = I_n$. It follows from (3.8) that

$$N(x) = E(x,1)\begin{pmatrix} \cos\rho J\tilde{\beta}^t J & -\sin\rho J\tilde{\beta}^t J \\ \sin\rho I_n & \cos\rho I_n \end{pmatrix} \tag{3.13}$$

where $E(x,1)$ and $\tilde{\beta}$ are as in (3.11) and (3.12), respectively. According to lemma 2.2 and theorem 3.6, one knows that the last volume of (3.13) gives the flat time-like manifold in S^{2n-1}_{2q}.

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